Convexity of reduced energy and mass angular momentum inequalities

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Abstract

In this paper, we extend the work in [7][4][3][6]. We weaken the asymptotic conditions on the second fundamental form, and we also give an L^6−norm bound for the difference between general data and Extreme Kerr data or Extreme Kerr-Newman data by proving convexity of the renormalized Dirichlet energy when the target has non-positive curvature. In particular, we give the first proof of the strict mass/angular momentum/charge inequality for axisymmetric Einstein/Maxwell data which is not identical with the extreme Kerr-Newman solution.

1 Introduction

An interesting question about solutions of the Einstein equations is whether the angular momentum (and charge for the Einstein/Maxwell case) can be bounded by the mass for physically reasonable solutions. This is true for the Kerr and Kerr-Newman black hole solutions which are stationary. For dynamical, axisymmetric solutions some general results have been obtained, first by S. Dain [7] and later by other authors [4][3][6] over the past several years. In this paper we introduce a new method for obtaining such inequalities which is technically simpler and which provides sharper results in many cases. We apply this method to both the vacuum black hole case and to the Einstein/Maxwell black hole case. An interesting feature of our method is that it provides a quantitative lower bound on the gap in the inequality in terms of an L^6 measure of the distance between the dynamical solution and the comparison stationary solution. As such it readily handles the borderline case, and provides an extremal characterization of the Kerr and Kerr-Newman solutions. In this paper we deal with the reduction of the initial data to a mapping and we state our theorems in terms of the mapping. For the corresponding statements in terms of physical quantities we refer to Theorem 1.1 of [4] for the vacuum case and to Theorem 1.1 of [6] for the Einstein/Maxwell case.

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It is well known that the Dirichlet energy for mappings from compact manifolds into negatively curved Riemannian manifolds has a strong convexity property along geodesic deformations [10]. Here we will prove a similar convexity result for the normalized Dirichlet energy of certain singular mappings to negatively curved Riemannian manifold arising from mathematical general relativity (see [7][11][4]). We will use this convexity to show that singular harmonic maps are unique in a class of maps with finite reduced energy and the same asymptotic singular behavior. Moreover, we can control the L^6 norm of the distance between any such map and the singular harmonic map by the reduced energy gap.

On $\mathbb{R}^3$, we use (ρ, φ, z) to denote cylindrical coordinates, and (r, θ, ϕ) to denote spherical coordinates. We use Γ to denote the $z-$axis which is given by $\{\rho \equiv 0\}$. We define g by

$$g = 2\log\rho, \tag{1.1}$$

and note that g is the potential of a uniform charge distribution on Γ. In particular g is harmonic on $\mathbb{R}^3 \setminus \Gamma$. Now we are interested in the mapping $(X, Y) : \Omega \subset \mathbb{R}^3 \to \mathbb{H}^2$, where $\mathbb{H}^2 = \{(X, Y) \in \mathbb{R}^2, X > 0\}$ is the hyperbolic right half plane with metric $ds^2_{-1} = \frac{dX^2+dY^2}{X^2}$. Since $X > 0$, we can rewrite X as $X = e^{g+x}$, or equivalently

$$x = \log X - g. \tag{1.2}$$

We are interested in the following functional discussed in [7].

$$\mathcal{M}_\Omega(x, Y) = \int_\Omega |\partial x|^2 + e^{-2g-2x}|\partial Y|^2 d\mu. \tag{1.3}$$

We denote $\mathcal{M}(x, Y) = \mathcal{M}_{\mathbb{R}^3}(x, Y)$. The motivation to study this functional is that the *extreme Kerr Solution* of the Einstein vacuum equations gives rise to a local critical point of the above functional. The extreme Kerr solution corresponds to the map (X_0, Y_0), or equivalently (x_0, Y_0) where $x_0 = \log X_0 - g$, which in spherical coordinates, is given by (see [7])

$$X_0 = \big(\tilde{r}^2 + |J| + \frac{2|J|^{3/2}\tilde{r}\sin^2\theta}{\Sigma}\big)\sin^2\theta, \quad Y_0 = 2J(\cos^3\theta - 3\cos\theta) - \frac{2J^2\cos\theta\sin^4\theta}{\Sigma}, \tag{1.4}$$

and

$$\tilde{r} = r + \sqrt{|J|}, \quad \Sigma = \tilde{r}^2 + |J|\cos^2\theta, \tag{1.5}$$

where the number J corresponds to the angular momentum of the spacetime corresponding to (X_0, Y_0).

Now we are interested in the class of (x, Y) such that functional $\mathcal{M}$ in equation (1.3) is well-defined, finite and physically corresponds to an axisymmetric initial data set for the vacuum Einstein equations[1]. In fact, we are interested in a class of data which can be written

[1] We refer this physical background to [7] and [8].

as variations of the Kerr solutions. Denote

$$x = x_0 + \alpha, \quad Y = Y_0 + y. \tag{1.6}$$

Let $\alpha \in H^1(\mathbb{R}^3)$, which is the completion of $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ under the norm

$$\|\alpha\|_1 = \big(\int_{\mathbb{R}^3} |\partial\alpha|^2 d\mu\big)^{1/2}, \tag{1.7}$$

and $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, which is the completion of $C_c^\infty(\mathbb{R}^3 \setminus \Gamma)$ under the norm

$$\|y\|_{1,X_0} = \big(\int_{\mathbb{R}^3} X_0^{-2}|\partial y|^2 d\mu\big)^{1/2}. \tag{1.8}$$

Here $d\mu$ denotes the Euclidean volume measure.

We will give a simplified proof of a strengthening of Theorem 1.2 of [7].

Theorem 1.1. *The functional $\mathcal{M}(x, Y)$ achieves a global minimum at the Extreme Kerr solution (x_0, Y_0) over all $\{x = x_0 + \alpha, Y = Y_0 + y\}$, where $\alpha \in H^1(\mathbb{R}^3)$ with $\alpha_- = \inf\{0, \alpha\} \in L^\infty(\mathbb{R}^3)$, and $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, that is, for any such (x, Y)*

$$\mathcal{M}(x, Y) \geq \mathcal{M}(x_0, Y_0). \tag{1.9}$$

Furthermore, we have the following gap bound,

$$\mathcal{M}(x, Y) - \mathcal{M}(x_0, Y_0) \geq C\{\int_{\mathbb{R}^3} d_{-1}^6\big((X, Y), (X_0, Y_0)d\mu\big)\}^{1/3} \tag{1.10}$$

where $d_{-1}(\cdot, \cdot)$ is the distance function on $\mathbb{H}^2$.

Remark 1.2. *Here the condition $\alpha_- \in L^\infty$ is needed to insure that $\mathcal{M}(x, Y)$ to be finite for $y \in H^1_{0,X_0}$. We do not need the L^∞ condition for $X_0^{-1}y$ which is assumed in Theorem 1.2 of [7], since we do not need to construct a minimizer of $\mathcal{M}$ in our proof.*

In [2], P. Chruściel generalized the class of axially symmetric initial data which admit a representation as a mapping to $\mathbb{H}^2$ and extended a theorem of D. Brill [1] to prove the positive mass theorem for data in this class. The mass/angular momentum inequality for this class was obtained by P. Chruściel, Y. Y. Li, and G. Weinstein [4]. In Section 4 we extend our method to recover their theorem in a stronger form including the gap estimate. This is done in Theorem 4.2. In addition to obtaining the L^6 lower bound for the gap, we weaken the asymptotic assumptions, requiring the second fundamental form h to decay strictly faster than $r^{-3/2}$ while the results of [4] require decay strictly faster than $r^{-5/2}$.

In Section 5 we apply our method to the case of Einstein/Maxwell black hole data. In this case the target manifold for the associated mapping is the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$

(four real dimensions). In Theorem 5.4 we give an extension of Theorem 1.1 to bound the gap in the reduced energy between a general map to $\mathbb{H}^2_{\mathbb{C}}$ in an appropriate asymptotic class (see (5.9)) and the harmonic map corresponding to the extremal Kerr-Neuman solution. In order to prove mass/angular momentum inequalities for black hole Einstein/Maxwell initial data, we extend our method in Section 6 to cover a class of initial data introduced by Chruściel and J. Costa [3], [6]. This requires a careful examination of the asymptotic conditions which is given in 6.1. The main theorem extending the results of [3] and [6] is Theorem 6.1. Our theorem includes a lower bound on the gap and therefore also implies the borderline case which gives a characterization of the Kerr-Newman solution. This does not appear to follow from [3] and [6].

2 Convexity for $\mathcal{M}$

The motivation to study convexity properties of $\mathcal{M}$ comes from the relation between $\mathcal{M}$ and the Dirichlet energy E, which is defined for $(X, Y) : \mathbb{R}^3 \to \mathbb{H}^2$ by

$$E(X,Y) = \int_{\mathbb{R}^3} \frac{|\partial X|^2 + |\partial Y|^2}{X^2} d\mu. \tag{2.1}$$

Here E is just the standard harmonic map energy[2] for mapping $(X, Y) : \mathbb{R}^3 \to \mathbb{H}^2$.

2.1 Convexity of the Dirichlet energy

Now let us first discuss a general result. Let (M, g) be a general n dimensional Riemannian manifold, and $\Omega \subset (M, g)$ an open subset with or without boundary. Let (N, h) be a target Riemannian manifold, and $u_0,\ u_1 : \Omega \to (N, h)$ be C^2 mappings. Now connect them by a C^2 family of mappings $F : \Omega \times [0, 1] \to (N, h)$. We denote the energy restricted to maps on Ω by E_Ω. We let F_t denote the map with t fixed, and we consider the second variation of the energy[3] of F_t. Denote the variational vector field by $V = F_*(\frac{\partial}{\partial t})$, then we have the **second variation formula**:

$$\begin{aligned} \frac{d^2}{dt^2} E_\Omega(F_t) = 2 \int_\Omega \Big[\sum_{\alpha=1}^n \|\nabla^N_{(F_t)_*(e_\alpha)} V\|_h^2 \\ - \sum_{\alpha=1}^n R^N(V, (F_t)_*(e_\alpha), V, (F_t)_*(e_\alpha)) - div_{F(M)}(\nabla^N_V V)\Big] dvol_M, \end{aligned} \tag{2.2}$$

where $\{e_\alpha\}_{\alpha=1}^n$ is a local orthonormal basis on (Ω, g). So if the target manifold (N, h) has non-positive sectional curvature, then the second term in the above integral is non-negative.

[2] See definition and properties in [10]

[3] The results went back to Section 3 of [10].

If we can choose F_t to be a geodesic deformation, i.e $F_t(x) : [0,1] \to (N,h)$ is a geodesic for any fixed $x \in \Omega$, then we know that $\nabla^N_V V \equiv 0$, so the last term in the above integral is zero. So we get $\frac{d^2}{dt^2} E_\Omega(F_t) \geq 0$, which is the convexity for the Dirichlet energy under geodesic deformations.

Moreover, we have a refined estimate. In the second variation formula (2.2), the third term in the integrand is zero, and the second term is nonnegative. To deal with the first term, we will use the following Kato inequality,

Lemma 2.1. *If e and V are two tangent vector fields on (N,h), then*

$$\|\nabla_e V\|_h \geq |\nabla_e \|V\|_h|. \tag{2.3}$$

Proof. We have

$$\nabla_e \|V\|_h = \frac{\langle \nabla_e V, V\rangle_h}{\|V\|_h},$$

so by the Cauchy-Schwartz inequality, we get the desired result. $\square$

Applying the above result to the first term in equation (2.2),

$$\begin{aligned}\sum_{\alpha=1}^n \|\nabla^N_{(F_t)_* e_\alpha} V\|_h^2 &\geq \sum_{\alpha=1}^n |\nabla^N_{(F_t)_* e_\alpha} \|V\|_h|^2 \\ &= \sum_{\alpha=1}^n |\nabla^M_{e_\alpha} (\|V\|_h \circ F_t)|^2.\end{aligned}$$

Since F_t is chosen to be a geodesic deformation, we know that

$$\|V\|_h(F_t(x)) = dist_h(F_0(x), F_1(x)) = dist_h(u_0(x), u_1(x)),$$

where $dist_h$ is the distance function of (N,h). Now putting this into equation (2.2), we have the **refined second variation formula**:

$$\frac{d^2}{dt^2} E_\Omega(F_t) \geq 2 \int_\Omega \|\nabla dist_h(u_0,u_1)\|_g^2 dvol_M. \tag{2.4}$$

If u_0 is a harmonic map, by integrating the above inequality twice with respect to the variable t, we can get an estimate of the L^2 norm of the gradient of the distance function $dist_h(u_0,u_1)$ by the energy gap.

2.2 Singular case

Now we will apply the same idea to our functional $\mathcal{M}$ under geodesic deformations. The first observation concerns the relation between $\mathcal{M}$ and E. Consider a compact open domain

$\Omega \subset \mathbb{R}^3 \setminus \Gamma$ and put condition (1.2) into equation (2.1). By an integration by parts argument based on the fact that g is harmonic, we get[4]

$$E_\Omega(X,Y) = \mathcal{M}_\Omega(x,Y) + \int_{\partial\Omega} \frac{\partial g}{\partial n}(g+2x)d\sigma, \tag{2.5}$$

where $\mathcal{M}_\Omega$ is the functional $\mathcal{M}$ restricted to domain Ω, n is the unit outer normal of $\partial\Omega$, and $d\sigma$ the area element of $\partial\Omega$. Since E and $\mathcal{M}$ only differ by a boundary integral, they must have the same critical points and thus we call $\mathcal{M}$ the *reduced energy*. In fact, $\mathcal{M}$ is a regularization of E in this special case since we are removing the infinite term $\int |\partial g|^2$ from E.

Now we obtain a convexity result for $\mathcal{M}_\Omega$. We first choose our compact domain Ω as an annulus region $A_{R,\epsilon} = B_R \setminus B_\epsilon$, where B_R denotes the Euclidean ball of radius R in $\mathbb{R}^3$. Denote $\Omega_{R,\epsilon} = A_{R,\epsilon} \setminus \mathcal{C}_\epsilon$ where $\mathcal{C}_\epsilon = \{\rho \leq \epsilon\}$ is the cylinder centered on the z axis Γ of radius ϵ. The definition of $H^1(\mathbb{R}^3)$ and $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$ motivate us to first consider functions $\alpha \in C_c^\infty(A_{R,\epsilon})$ and $y \in C_c^\infty(\Omega_{R,\epsilon})$, with $X = e^{g+x_0+\alpha}$ and $Y = Y_0 + y$. Now consider a geodesic deformation

$$F : A_{R,\epsilon} \times [0,1] \to \mathbb{H}^2,$$

with $F_0 = (X_0, Y_0)$ and $F_1 = (X, Y)$. Denote $F_t = (X_t, Y_t)$, $x_t = \log X_t - g$, and $y_t = Y_t - Y_0$.

Now we make an important observation that reduces the computational difficulty substantially. Since $y \in C_c^\infty(\Omega_{R,\epsilon})$, we know that on a neighborhood of $\mathcal{C}_\epsilon \cap A_{R,\epsilon}$, $Y \equiv Y_0$, and $X = X_0 e^\alpha$. By basic hyperbolic geometry, we know that the geodesic from (X_0, Y_0) to $(X = X_0 e^\alpha, Y = Y_0)$ is given by

$$X_t = X_0 e^{t\alpha}, \quad Y_t = Y_0. \tag{2.6}$$

By using equation (1.2), we have that on a neighborhood of $\mathcal{C}_\epsilon \cap A_{R,\epsilon}$,

$$x_t = x_0 + t\alpha. \tag{2.7}$$

Now let us compute the second variation of the reduced energy $\mathcal{M}_{A_{R,\epsilon}}$

$$\frac{d^2}{dt^2}\mathcal{M}_{A_{R,\epsilon}}(x_t, Y_t) = \frac{d^2}{dt^2}\mathcal{M}_{\Omega_{R,\epsilon}}(x_t, Y_t) + \frac{d^2}{dt^2}\mathcal{M}_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}(x_t, Y_t).$$

[4] This is also given by equation (66) of [7].

For the first term, we use equation (2.5)

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{M}_{\Omega_{R,\epsilon}}(x_t,Y_t) &= \frac{d^2}{dt^2}E_{\Omega_{R,\epsilon}}(X_t,Y_t) - \frac{d^2}{dt^2}\int_{\partial\Omega_{R,\epsilon}}\frac{\partial g}{\partial n}(g+2x_t)d\sigma;\\
&= \frac{d^2}{dt^2}E_{\Omega_{R,\epsilon}}(X_t,Y_t) - 2\frac{d^2}{dt^2}\int_{\partial\Omega_{R,\epsilon}\cap\mathcal{C}_{R,\epsilon}}\frac{\partial g}{\partial n}x_t d\sigma;\\
&= \frac{d^2}{dt^2}E_{\Omega_{R,\epsilon}}(X_t,Y_t) - 2\frac{d^2}{dt^2}\int_{\partial\Omega_{R,\epsilon}\cap\mathcal{C}_{R,\epsilon}}\frac{\partial g}{\partial n}(x_0+t\alpha)d\sigma; \qquad (2.8)\\
&= \frac{d^2}{dt^2}E_{\Omega_{R,\epsilon}}(X_t,Y_t)\\
&\geq 2\int_{\Omega_{R,\epsilon}}|\nabla dist_{-1}\big((X,Y),(X_0,Y_0)\big)|^2 d\mu.
\end{aligned}$$

Here $dist_{-1}$ is the distance function on the hyperbolic plane $\mathbb{H}_{-1}$. The second " $=$ " is because that $x_t \equiv x_0$ near $\partial A_{R,\epsilon}\cap\Omega_{R,\epsilon}$ since α is compactly supported in $A_{R,\epsilon}$. The third " $=$ " is given by equation (2.7). The last " $=$ " is because the second term there is linear in t. The last inequality " $\geq$ " comes from the convexity of the harmonic energy (2.4) along geodesic paths.

Now we deal with the second part by direct calculation

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{M}_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}(x_t,Y_t) &= \frac{d^2}{dt^2}\int_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}|\nabla x_t|^2 + e^{-2g-2x_t}|\nabla Y_t|^2 d\mu\\
&= \frac{d^2}{dt^2}\int_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}|\nabla(x_0+t\alpha)|^2 + e^{-2g-2(x_0+t\alpha)}|\nabla Y_0|^2 d\mu\\
&= \int_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}2|\nabla\alpha|^2 + 4\alpha^2 e^{-2g-2(x_0+t\alpha)}|\nabla Y_0|^2 d\mu \qquad (2.9)\\
&\geq \int_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}2|\nabla\alpha|^2 d\mu\\
&= 2\int_{A_{R,\epsilon}\cap\mathcal{C}_\epsilon}|\nabla dist_{-1}\big((X,Y),(X_0,Y_0)\big)|^2 d\mu.
\end{aligned}$$

The second " $=$ " comes from equation (2.7) again. The last " $=$ " follows from the equation (2.6) on $A_{R,\epsilon}\cap\mathcal{C}_\epsilon$ and the fact that the distance $d_{-1}\big((X,Y),(X_0,Y_0)\big) = \alpha$.

Remark 2.2. *We can put $\frac{d^2}{dt^2}$ into the integral because that the integrands are all uniformly integrable.*

Now combining the above inequalities, we get the desired convexity under geodesic deformation,

Lemma 2.3. *With (X_0,Y_0) and (X,Y) as above we have*

$$\frac{d^2}{dt^2}\mathcal{M}_{A_{R,\epsilon}}(x_t,Y_t) \geq 2\int_{A_{R,\epsilon}}|\nabla d_{-1}\big((X,Y),(X_0,Y_0)\big)|^2 d\mu. \qquad (2.10)$$

3 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1.

Proof. For $\alpha \in H^1(\mathbb{R}^3)$, $\alpha_- = \inf\{0, \alpha\} \in L^\infty(\mathbb{R}^3)$, and $y \in H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, by the definition of $H^1(\mathbb{R}^3)$ and $H^1_{0,X_0}(\mathbb{R}^3 \setminus \Gamma)$, we can choose two sequences of mappings $\{\alpha_n \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})\}_{n=1}^\infty$ and $\{y_n \in C_c^\infty(\mathbb{R}^3 \setminus \Gamma)\}_{n=1}^\infty$, such that[5],

$$\|\alpha - \alpha_n\|_1 \to 0, \quad \|y - y_n\|_{1,X_0} \to 0. \tag{3.1}$$

It is easy to see that

$$\mathcal{M}(x_n, Y_n) \to \mathcal{M}(x, Y), \tag{3.2}$$

where $x_n = x_0 + \alpha_n$, $Y_n = Y_0 + y_n$, and (x, Y) is given in Theorem 1.1. We can further assume that there exist two sequences of positive numbers $\{R_n \to \infty\}_{n=1}^\infty$ and $\{\epsilon_n \to 0\}_{n=1}^\infty$, such that $\alpha_n \in C_c^\infty(A_{R_n,\epsilon_n})$, and $y_n \in C_c^\infty(\Omega_{R_n,\epsilon_n})$.

Now we would like to use the argument in the proof of uniqueness of harmonic mappings when the ambient manifold is negatively curved[6]. For fixed n, we focus on the region A_{R_n,ϵ_n} and Ω_{R_n,ϵ_n}. We will discard the sub-index n in the following argument. There is a geodesic deformation $F_t : A_{R,\epsilon} \to \mathbb{H}^2$ from (X_0, Y_0) to $(X = X_0 e^\alpha, Y = Y_0 + y)$. We know that $\mathcal{M}_{A_{R,\epsilon}}(F_t)$ is a convex function from above. Since (X_0, Y_0) is harmonic on $\mathbb{R}^3 \setminus \Gamma$, we will show that (x_0, Y_0) is critical point of the reduced functional $\mathcal{M}_{A_{R,\epsilon}}$. In fact, we have[7]:

$$\triangle \log X_0 = -\frac{|\partial Y_0|^2}{X_0^2}, \tag{3.3}$$

$$\triangle Y_0 = 2\frac{\langle \partial Y_0, \partial X_0 \rangle}{X_0}. \tag{3.4}$$

Lemma 3.1. *At $t = 0$ we have*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{M}_{A_{R,\epsilon}}(F_t) = 0. \tag{3.5}$$

Proof. We compute

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathcal{M}_{A_{R,\epsilon}}(x_t, Y_t) &= \frac{d}{dt}\Big|_{t=0} \int_{A_{R,\epsilon}} |\partial x_t|^2 + e^{-2g-2x_t}|\partial Y_t|^2 d\mu \\
&= 2\int_{A_{R,\epsilon}} \langle \partial x_0, \partial x_0' \rangle - x_0' e^{-2g-2x_0}|\partial Y_0|^2 + e^{-2g-2x_0}\langle \partial Y_0, \partial Y_0' \rangle d\mu.
\end{aligned}$$

[5]See equation (1.7) and (1.8)

[6]See Section 3 of [10]

[7]See equations (70)(71) in [7].

Here we put the $\frac{d}{dt}$ into the integral in the second " $=$ " since the integrand is uniformly integrable.

Taking $\lambda \ll \epsilon$, we separate $A_{R,\epsilon}$ into two parts $A_{R,\epsilon} \setminus \mathcal{C}_\lambda$ and $A_{R,\epsilon} \cap \mathcal{C}_\lambda$. Using that (X_0, Y_0) satisfies the Euler-Lagrange equations(3.3)(3.4) for $\mathcal{M}$ to do integration by parts on $A_{R,\epsilon} \setminus \mathcal{C}_\lambda$ where all functions are regular, and noticing the fact that $Y_0' \equiv 0$ near $\mathcal{C}_\lambda$, we have

$$\frac{d}{dt}\Big|_{t=0} \mathcal{M}_{A_{R,\epsilon}}(x_t, Y_t) = 2\int_{\{\rho=\lambda\}\cap A_{R,\epsilon}} \frac{\partial x_0}{\partial n} \cdot \alpha d\sigma + 2\int_{A_{R,\epsilon}\cap \mathcal{C}_\lambda} \langle \partial x_0, \partial \alpha\rangle - \alpha e^{-2g-2x_0}|\partial Y_0|^2 d\mu.$$

The integrals above converge to 0 as $\lambda \to 0$ since α and $\frac{\partial x_0}{\partial n}$ are bounded and all the other integrands are uniformly integrable on $A_{R,\epsilon} \cap \mathcal{C}_\lambda$. $\square$

Let us return to the proof of Theorem 1.1. Integrating inequality (2.10) with respect to t once, and using the fact that $\frac{d}{dt}\big|_{t=0}\mathcal{M}_{A_{R,\epsilon}}(x_t, Y_t) = 0$ we get,

$$\frac{d}{dt}\mathcal{M}_{A_{R,\epsilon}}(x_t, Y_t) \geq 2t \int_{A_{R,\epsilon}} |\partial d_{-1}\big((X,Y),(X_0,Y_0)\big)|^2 d\mu.$$

Integrating with respect to t again, we get

$$\mathcal{M}(x,Y) - \mathcal{M}(x_0,Y_0) \geq \int_{A_{R,\epsilon}} |\partial d_{-1}\big((X,Y),(X_0,Y_0)\big)|^2 d\mu.$$

Since the difference between (x, Y) and (x_0, Y_0) is now restricted to a compact domain B_R, we can apply the scale invariant Sobolev inequality(see Theorem 1 on page 263 in [9]) to get,

$$\mathcal{M}(x,Y) - \mathcal{M}(x_0,Y_0) \geq \frac{1}{C}\Big(\int_{A_{R,\epsilon}} |d_{-1}\big((X,Y),(X_0,Y_0)\big)|^6 d\mu\Big)^{\frac{1}{3}}. \tag{3.6}$$

In order to extend the above inequality to the general case $\alpha = x - x_0 \in H^1(\mathbb{R}^3)$ and $y = Y - Y_0 \in H_0^1(\mathbb{R}^3 \setminus \Gamma)$, we first use the compactly supported approximating sequence $\{(\alpha_n, y_n)\}$ (3.1) into (3.6). By basic hyperbolic geometry

$$\begin{aligned}
d_{-1}\big((X,Y),(X_n,Y_n)\big) &= d_{-1}\big((X_0e^\alpha, Y_0+y),(X_0e^{\alpha_n}, Y_0+y_n)\big)\\
&\leq d_{-1}\big((X_0e^\alpha, Y_0+y),(X_0e^{\alpha}, Y_0+y_n)\big) + d_{-1}\big((X_0e^\alpha, Y_0+y_n),(X_0e^{\alpha_n}, Y_0+y_n)\big)\\
&= e^{-\alpha}\frac{|y-y_n|}{X_0} + |\alpha - \alpha_n| \to 0, \quad \text{almost everywhere in } \mathbb{R}^3,
\end{aligned}$$

since $\alpha_- \in L^\infty$. Hence

$$|d_{-1}\big((X_n,Y_n),(X_0,Y_0)\big) - d_{-1}\big((X,Y),(X_0,Y_0)\big)| \to 0, \text{ almost everywhere in } \mathbb{R}^3.$$

Using (3.2) and Fatou's lemma to take the limit, we have proven (1.10). $\square$

4 Extension to Chruściel data

In this section we apply the convexity argument to the class of initial data defined in [2][4]. We first review the conditions on this data.

4.1 Review of [2][4]

Let us briefly review Chruściel's reduction[2]. Let (M, g) be a *3-dimensional simply connected asymptotically flat* manifold, say with two ends, such that each end M_{ext} is diffeomorphic to $\mathbb{R}^3 \setminus B(R)$. Assume that there are coordinates on $\mathbb{R}^3 \setminus B(R)$ such that in these coordinates the metric g satisfies,

$$g_{ij} - \delta_{ij} = o_k(r^{-1/2}), \quad k \geq 5. \tag{4.1}$$

Assume (M, g) is axisymmetric, i.e. there exists a killing vector field η with complete periodic orbits, such that $\mathcal{L}_\eta g = 0$, then by Theorem 2.9 in [2], $M \simeq \mathbb{R}^3 \setminus \{0\}$, where one end is at ∞ and the other at the origin 0, and the metric g can be written

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\varphi + \rho B_\rho d\rho + A_z dz)^2, \tag{4.2}$$

where (ρ, φ, z) are cylindrical coordinates of $\mathbb{R}^3$, and all functions are φ independent. Furthermore, in these coordinates we have

$$\eta = \partial_\varphi, \tag{4.3}$$

and

$$U = o_{k-3}(r^{-1/2}), \quad r \to \infty, \tag{4.4}$$

$$\alpha = o_{k-4}(r^{-1/2}), \quad r \to \infty, \tag{4.5}$$

$$U = 2 \log r + o_{k-4}(r^{1/2}), \quad r \to 0, \tag{4.6}$$

$$\alpha = o_{k-4}(r^{1/2}), \quad r \to 0. \tag{4.7}$$

Now let (M, g, h) be a *simply connected, asymptotically flat, maximal, axisymmetric, vacuum initial data set* for the Einstein equations. We assume (M, g) is as above, and we assume the asymptotic decay for h on each end M_{ext},

$$|h|_g = O_{k-1}(r^{-\lambda}), \ r \to \infty, \lambda > 3/2. \tag{4.8}$$

Remark 4.1. *Note that our decay rate for h is faster than $-3/2$, while in [4], they require the decay rate to be faster than $-5/2$.*

Now the vacuum constraint equation for (g, h) and the maximal condition $tr_g h = 0$ imply $*_g(i_\eta h \wedge \eta)$ is closed[8], which is then exact since $\pi_1(M) = 0$, so there exists a function w, such that,

$$dw = *_g(i_\eta h \wedge \eta), \tag{4.9}$$

where $*_g$ is the Hodge star operator for g. In our notation in Section 1

$$U = -\frac{1}{2}x, \quad w = \frac{1}{2}Y. \tag{4.10}$$

It is obvious that $dw \equiv 0$ on the axis $\Gamma = \{\rho = 0,\ z \neq 0\}$ since $\eta \equiv 0$ there. We will normalize w so that,

$$w|_{\mathcal{A}_i} = w_i, \tag{4.11}$$

where $\mathcal{A}_1 = \{\rho = 0, z < 0\}$, $\mathcal{A}_2 = \{\rho = 0, z > 0\}$ are the two parts of the axis Γ, and w_i corresponds to the value of Extreme Kerr solution (1.4) on $\mathcal{A}_i$.

Now by the decay (4.1)(4.8) of (g, h) and the definition of dw (4.9), we can derive the decay rate of dw at infinity,

$$|Dw|_\delta \leq C\rho^2 r^{-\lambda},\ r \to \infty. \tag{4.12}$$

By an inversion formula $x \to \frac{x}{|x|^2}$, which is done in (2.31)(2.32) in [4], we can get the blow up rate of dw near origin,

$$|Dw|_\delta \leq C'\rho^2 r^{\lambda-6},\ r \to 0. \tag{4.13}$$

Using (4.9) and (4.2) we have decay estimates of dw near the axis away from 0 and ∞,

$$|Dw|_\delta \leq C(\delta)\rho^2,\ \rho \to 0,\ \delta \leq r \leq 1/\delta, \tag{4.14}$$

where $C(\delta)$ is a constant depending on δ.

From (2.10) in [4], we have a bound for the ADM mass m of (M, g, h) when $k \geq 6$,

$$m \geq \frac{1}{8\pi}\int_{\mathbb{R}^3} \left[|DU|^2 + \frac{e^{4U}}{\rho^4}|Dw|^2\right]dx. \tag{4.15}$$

Now we will apply the convexity argument to the functional

$$\mathcal{I}(U, w) := \int_{\mathbb{R}^3} \left[|DU|^2 + \frac{e^{4U}}{\rho^4}|Dw|^2\right]dx. \tag{4.16}$$

Theorem 4.2. *For $k \geq 6$, $\mathcal{I}(U, w)$ is bounded from below by the corresponding value of the Extreme Kerr data(1.4), i.e. $\mathcal{I}_0 = \mathcal{I}(U_0, w_0)$, among all data $\{(U, w)\}$ satisfying (4.4)(4.6)(4.12) (4.13)(4.14) and (4.11), i.e.*

$$\mathcal{I}(U, w) \geq \mathcal{I}(U_0, w_0). \tag{4.17}$$

[8]See Section 2 of [7].

Moreover, we have the gap bound,

$$\mathcal{I}(U,w) - \mathcal{I}(U_0,w_0) \geq C\{\int_{\mathbb{R}^3} d_{-1}^6\big((U,w),(U_0,w_0)\big)dx\}^{1/3}, \tag{4.18}$$

where $d_{-1}\big((U,w),(U_0,w_0)\big)$ *is the distance between* $(\rho^2 e^{-2U}, 2w)$ *and* $(\rho^2 e^{-2U_9}, 2w_0)$ *with respect to the hyperbolic metric* ds_{-1}^2.

Remark 4.3. *Let us say a few words about the integrability of* $\mathcal{I}(U,w)$ *under conditions (4.4)(4.6)(4.12) and (4.13). In fact, near* ∞, $|DU|^2 = o(r^{-3})$ *is integrable, and* $\frac{e^{4U}}{\rho^4}|Dw|^2 = O(r^{-2\lambda})$ *is also integrable, when* $\lambda > 3/2$. *Near the singularity* 0, $|DU|^2 = O(r^{-2})$ *is integrable, and* $\frac{e^{4U}}{\rho^4}|Dw|^2 = O(\frac{r^8}{\rho^4}\cdot\rho^4 r^{2\lambda-12}) = O(r^{2\lambda-4})$ *which is integrable only when* $\lambda > 1/2$.

For the extreme Kerr solution (U_0,w_0), *the blow up rate at the origin* 0 *and decay rate at* ∞ *are*[9]*:*

$$U_0 = \log r + C, \quad |Dw_0|_\delta \leq C\frac{\rho^2}{r^3}: \quad r\to 0. \tag{4.19}$$

$$|Dw_0|_\delta \leq C\frac{\rho^2}{r^3}: \quad r\to\infty. \tag{4.20}$$

So the integrability of $\mathcal{I}(U_0,w_0)$ *follows as above.*

4.2 Cut and paste argument

Given data (U,w) as in Theorem 4.2, the idea is that $\mathcal{I}(U,w)$ can be approximated by cutting and pasting (U,w) to (U_0,w_0) near ∞, and then cutting and pasting w to w_0 near 0 and the axis Γ. An idea of this type is used in [4], but we take a different approximation here.

Propostion 4.4. *Under conditions (4.4)(4.6)(4.12)(4.13)(4.14) and (4.11) for* (U,w), *for any small* $c_0 > 0$ *we can find* $(U_\delta, w_{\delta,\epsilon})$ *for small* $\epsilon \ll \delta \ll 1$, *such that:*

$$U_\delta \equiv U,\ r<1/\delta;\ w_{\delta,\epsilon}\equiv w,\ \rho>\sqrt{\epsilon},\ 2\delta<r<1/\delta,$$

$$(U_\delta, w_{\delta,\epsilon}) = (U_0,w_0),\ r>2/\delta;\ w_\delta\equiv w_0,\ x\in B_\delta\cup\mathcal{C}_{\delta,\epsilon},$$

where $\mathcal{C}_{\delta,\epsilon}$ *is defined in (4.24), and*

$$|\mathcal{I}(U,w) - \mathcal{I}(U_\delta, w_{\delta,\epsilon})| < c_0.$$

The proof is a combination of the following three lemmas. Let us define a family of smooth functions $\varphi_\delta^1 \in C_c^\infty(\mathbb{R}^3)$:

$$\varphi_\delta^1(r)\begin{cases} = 1 & \text{if } r\leq 1/\delta \\ |D\varphi_\delta^1| \leq 2\delta & \text{if } 1/\delta < r < 2/\delta \\ = 0 & \text{if } r \geq 2/\delta. \end{cases} \tag{4.21}$$

[9] See Appendix A of [4].

Now define

$$U^1_\delta = U_0 + \varphi^1_\delta(U - U_0), \quad w^1_\delta = w_0 + \varphi^1_\delta(U - U_0).$$

Then $(U^1_\delta, w^1_\delta) \equiv (U_0, w_0)$ outside $B_{2/\delta}$.

Lemma 4.5. *We have* $\lim_{\delta\to 0} \mathcal{I}(U^1_\delta, w^1_\delta) = \mathcal{I}(U, w)$.

Proof. We separate into three terms

$$\mathcal{I}(U^1_\delta, w^1_\delta) = \underbrace{\int_{r\le 1/\delta}}_{I_1} + \underbrace{\int_{1/\delta<r<2/\delta}}_{I_2} + \underbrace{\int_{r\ge 2/\delta}}_{I_3} \left[|DU^1_\delta|^2 + \frac{e^{4U^1_\delta}}{\rho^4}|Dw^1_\delta|^2\right] dx.$$

By the dominated convergence theorem(DCT[10]),

$$I_1 = \int_{r\le 1/\delta} [|DU|^2 + \frac{e^{4U}}{\rho^4}|Dw|^2] \to \mathcal{I}(U, w),$$

and

$$I_3 = \int_{r\ge 2/\delta} [|DU_0|^2 + \frac{e^{4U_0}}{\rho^4}|Dw_0|^2] dx \to 0.$$

$$I_2 = \underbrace{\int_{1/\delta<r<2/\delta} |DU^1_\delta|^2 dx}_{I_{21}} + \underbrace{\int_{1/\delta<r<2/\delta} \frac{e^{4U^1_\delta}}{\rho^4}|Dw^1_\delta|^2 dx}_{I_{22}},$$

where

$$I_{21} \le 2\int_{1/\delta<r<2/\delta} |DU|^2 + |DU_0|^2 + 2\int_{1/\delta<r<2/\delta} \underbrace{(U - U_0)^2}_{\sim o(r^{-1})} \underbrace{|D\varphi^1_\delta|^2}_{\le 4\delta^2} dx.$$

The first term converges to 0 by DCT and remark 4.3, and the second term is asymptotic to $o(1)$ since $r \sim \delta$ in this region, so it also converges to 0. We also have

$$I_{22} \le 4\int_{1/\delta<r<2/\delta} \frac{1}{\rho^4}(|Dw|^2 + |Dw_0|^2) + 4\int_{1/\delta<r<2/\delta} \frac{1}{\rho^4} \underbrace{(w - w_0)^2}_{\sim C\rho^6 r^{-2\lambda}} \underbrace{|D\varphi^1_\delta|^2}_{\le 4\delta^2} dx.$$

This is because both U and U_0 behave like $o(1)$ at infinity, so $e^{U^1_\delta}$ is bounded by 2 for δ small enough. The first term converges to 0 by DCT. The bound of $(w - w_0)$ comes from the fact that $(w - w_0)|_\Gamma \equiv 0$ and an integration of (4.12)(4.20) along a line perpendicular to the axis Γ. So the second term is asymptotic to $O(\delta^{2\lambda-3})$ since $r \sim \delta$, which converges to 0 when $\lambda > 3/2$. So we can get the limit by combining these results. $\square$

[10]We will abbreviate DCT as dominant convergence theorem in the follow.

Now we can first assume $U = U_0$ and $w = w_0$ outside a large ball B_R. Define a second family of smooth cutoff functions $\varphi_\delta \in C^\infty(\mathbb{R}^3)$,

$$\varphi_\delta(r) \begin{cases} = 0 & \text{if } r \leq \delta \\ |D\varphi_\delta| \leq 2/\delta & \text{if } \delta < r < 2\delta \\ = 1 & \text{if } r \geq 2\delta. \end{cases} \tag{4.22}$$

We let

$$w_\delta = w_0 + \varphi_\delta(w - w_0).$$

Then $w_\delta \equiv w_0$ inside the ball B_δ.

Lemma 4.6. *We have the result* $\lim_{\delta\to 0} \mathcal{I}(U, w_\delta) = \mathcal{I}(U, w)$.

Proof. We consider three terms

$$\mathcal{I}(U, w_\delta) = \underbrace{\int_{r\leq\delta}}_{I_1} + \underbrace{\int_{\delta<r<2\delta}}_{I_2} + \underbrace{\int_{r\geq 2\delta}}_{I_3} |DU|^2 + \frac{e^{4U}}{\rho^4}|Dw_\delta|^2 dx.$$

By DCT,

$$I_3 = \int_{r\geq 2\delta} |DU|^2 + \frac{e^{4U}}{\rho^4}|Dw|^2 \to \mathcal{I}(U, w).$$

On the other hand

$$I_1 = \int_{r\leq\delta} |DU|^2 + \frac{1}{\rho^4} \underbrace{e^{4U}}_{\sim r^8} \underbrace{|Dw_0|^2}_{\sim \frac{\rho^4}{r^6}} dx.$$

The first term converges to 0 by DCT. The second term, where we use (4.6)(4.19), is asymptotic to δ^5, hence converges to 0. To handle I_2 we estimate

$$\begin{aligned} I_2 \leq & \int_{\delta<r<2\delta} |DU|^2 + 2\frac{e^{4U}}{\rho^4}|Dw|^2 + 2\int_{\delta<r<2\delta} \frac{e^{4U}}{\rho^4}|Dw_0|^2 \\ & + 2\int_{\delta<r<2\delta} \frac{1}{\rho^4} \underbrace{e^{4U}}_{\sim r^8} \underbrace{(w - w_0)^2}_{\sim \rho^6 r^{2\lambda-12}} \underbrace{|D\varphi_\delta|^2}_{\leq 4/\delta^2} dx. \end{aligned}$$

The first term converges to 0 by DCT. The second term converges to 0 by the same argument as for I_1. The bound of $(w - w_0)$ comes from $(w - w_0)|_\Gamma \equiv 0$ and an integration of (4.13)(4.19) along a line perpendicular to the axis Γ. The last term is asymptotic to $O(\delta^{2\lambda-1})$ since $r \sim \delta$, which converges to 0. Combining these together, we get the limit. □

Remark 4.7. *The reason we can do this is because the blow-up rate($\rho^4 r^{-6}$) of $|Dw_0|^2$ is smaller than that($\rho^4 r^{2\lambda-12}$) of $|Dw|^2$ near the origin 0, while the decay rate (r^8) of e^{4U} is larger than that (r^4) of e^{4U_0}, so $|Dw_0|^2$ is also integrable with respect to $\frac{e^{4U}}{\rho^4}dx$ near the origin 0.*

Besides assuming $(U, w) \equiv (U_0, w_0)$ outside a large ball B_R, we can also assume $w \equiv w_0$ inside B_δ. Now define a third family of cutoff functions $\phi_\epsilon \in C^\infty(\mathbb{R}^3)$,

$$\phi_\epsilon(\rho) = \begin{cases} 0 & \text{if } \rho \leq \epsilon \\ \frac{\ln(\rho/\epsilon)}{\ln(\sqrt{\epsilon}/\epsilon)} & \text{if } \epsilon < \rho < \sqrt{\epsilon} \\ 1 & \text{if } \rho \geq \sqrt{\epsilon} \end{cases} \tag{4.23}$$

Define

$$w_\epsilon = w_0 + \phi_\epsilon(w - w_0).$$

Define the sets

$$\mathcal{C}_{\delta,\epsilon} = \{\rho \leq \epsilon\} \cap \{\delta \leq r \leq 2/\delta\}, \tag{4.24}$$

$$\mathcal{W}_{\delta,\epsilon} = \{\epsilon \leq \rho \leq \sqrt{\epsilon}\} \cap \{\delta \leq r \leq 2/\delta\}. \tag{4.25}$$

So we have $w_\epsilon \equiv w_0$ in $\mathcal{C}_{\delta,\epsilon} \cup B_\delta$.

Lemma 4.8. *We have the limit* $\lim_{\epsilon \to 0} \mathcal{I}(U, w_\epsilon) \to \mathcal{I}(U, w)$.

Proof. We consider three terms

$$\mathcal{I}(U, w_\epsilon) = \underbrace{\int_{\mathcal{C}_{\delta,\epsilon}}}_{I_1} + \underbrace{\int_{\mathcal{W}_{\delta,\epsilon}}}_{I_2} + \underbrace{\int_{\mathbb{R}^3 \setminus \{\mathcal{C}_{\delta,\epsilon} \cup \mathcal{W}_{\delta,\epsilon}\}}}_{I_3} |DU|^2 + \frac{e^{4U}}{\rho^4}|Dw_\epsilon|^2 dx.$$

By *DCT*, $I_3 \to \mathcal{I}(U, w)$.

$$I_1 = \int_{\mathcal{C}_{\delta,\epsilon}} |DU|^2 + \frac{e^{4U}}{\rho^4} \underbrace{|Dw_0|^2}_{\leq C\rho^4} dx.$$

The first term converges to 0 by DCT, while the bound $|Dw_0|_\delta$ come from (A.10) of [4]. The second term also converges to 0 by DCT. To handle I_2 we estimate

$$\begin{aligned} I_2 \leq & \int_{\mathcal{W}_{\delta,\epsilon}} |DU|^2 + 2\frac{e^{4U}}{\rho^4}|Dw|^2 + 2\int_{\mathcal{W}_{\delta,\epsilon}} \frac{e^{4U}}{\rho^4}|Dw_0|^2 \\ & + 2\int_{\mathcal{W}_{\delta,\epsilon}} \frac{e^{4U}}{\rho^4} \underbrace{(w - w_0)^2}_{\leq C\rho^4} \underbrace{|D\phi_\epsilon|^2}_{\sim 1/(\rho \ln \epsilon)^2} dx. \end{aligned}$$

The first two terms converge to 0 by DCT and the above argument as $\epsilon \to 0$. The bound of $(w - w_0)$ is gotten by integrating $\partial_\rho(w - w_0)$ along a line perpendicular to Γ with $(w - w_0)|_\Gamma \equiv 0$. So the last term is bounded by $C/|\ln \epsilon|$, which converges to 0 as $\epsilon \to 0$. We have completed the proof. $\square$

4.3 Convexity and gap inequality

As in the first section, we denote

$$U = U_0 + \alpha, \quad w = w_0 + y.$$

By Proposition 4.4, we can first assume (α, y) is compactly supported in $B_{2/\delta}$, and furthermore y is compactly supported in $\Omega_{\delta,\epsilon}$, where

$$\Omega_{\delta,\epsilon} = \{\delta < r < 2/\delta,\ \rho > \epsilon\}. \tag{4.26}$$

Denote

$$\mathcal{A}_{\delta,\epsilon} = B_{2/\delta} \setminus \Omega_{\delta,\epsilon}. \tag{4.27}$$

Now connect $(X = \rho^2 e^{-2U}, 2w = 2w_0 + 2y)$ to the Extreme Kerr data $(X_0 = \rho^2 e^{-2U_0}, Y_0 = 2w_0)$(1.4) by a geodesic family $(X_t, 2w_t)$ in $\mathbb{H}^2$. Let $U_t = -\frac{1}{2}\ln X_t + \log\rho$ and $y_t = w_t - w_0$. Hence $w_t \equiv w_0$ in a neighborhood of $\mathcal{A}_{\delta,\epsilon}$, so $U_t = U_0 + t\alpha$ in a neighborhood of $\mathcal{A}_{\delta,\epsilon}$ as discussed in Section 2. Then using the notation of Theorem 4.2 we have the following result.

Lemma 4.9. *We have*

$$\frac{d^2}{dt^2}\mathcal{I}(U_t, w_t) \geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla[d_{-1}\big((U,w),(U_0,w_0)\big)]|^2 dx. \tag{4.28}$$

Proof. We compute

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{I}(U_t, w_t) &= \frac{d^2}{dt^2}\mathcal{I}_{B_{2/\delta}}(U_t, w_t)\\ &= \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\Omega_{\delta,\epsilon}}(U_t, w_t)}_{I_1} + \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\mathcal{A}_{\delta,\epsilon}}(U_t, w_t)}_{I_2}.\end{aligned}$$

From equation (2.5) we have $E_\Omega(X, 2w) = 4\mathcal{I}_\Omega(U, w) + \int_{\partial\Omega}\frac{\partial g}{\partial n}(g - 4U)d\sigma$ on any compact domain Ω of $\mathbb{R}^3 \setminus \Gamma$. The first term is calculated as in (2.8):

$$\begin{aligned}I_1 &= \frac{1}{4}\frac{d^2}{dt^2}E_{\Omega_{\delta,\epsilon}}(X_t, 2w_t) + \frac{1}{4}\frac{d^2}{dt^2}\int_{\partial\Omega_{\delta,\epsilon}\cap\partial\mathcal{A}_{\delta,\epsilon}}\frac{\partial g}{\partial n}(g - 4(U_0 + t\alpha))d\sigma\\ &\geq \frac{1}{2}\int_{\Omega_{\delta,\epsilon}} |\nabla[d_{-1}\big((U,w),(U_0,w_0)\big)]|^2 dx.\end{aligned}$$

Using the fact that $d_{-1}\big((U,w),(U_0,w_0)\big) = 2|\alpha|$ on $\mathcal{A}_{\delta,\epsilon}$, the second term is calculated as:

$$\begin{aligned}I_2 &= \frac{d^2}{dt^2}\int_{\mathcal{A}_{\delta,\epsilon}} |D(U_0 + t\alpha)|^2 + \frac{1}{\rho^4}e^{4(U_0+t\alpha)}|Dw_0|^2 dx\\ &= 2\int_{\mathcal{A}_{\delta,\epsilon}} |D\alpha|^2 + 8\frac{1}{\rho^4}\alpha^2 e^{4(U_0+t\alpha)}|Dw_0|^2 dx\\ &\geq \frac{1}{2}\int_{\mathcal{A}_{\delta,\epsilon}} |D[d_{-1}\big((U,w),(U_0,w_0)\big)]|^2 dx.\end{aligned}$$

Now let us check the validity for putting $\frac{d^2}{dt^2}$ into the $\int_{\mathcal{A}_{\delta,\epsilon}}$. We need to show the integrand after the second " = " is uniformly integrable for all $t \in [0,1]$. The first term $\int_{\mathcal{A}_{\delta,\epsilon}} |D\alpha|^2 dx$ is integrable since both $U, U_0 \in H^1$. For the second term, let us separate $\mathcal{A}_{\delta,\epsilon} = B_\delta \cup \mathcal{C}_{\delta,\epsilon}$. Then on $\mathcal{C}_{\delta,\epsilon}$, $\frac{1}{\rho^4} \underbrace{\alpha^2}_{bounded} \underbrace{e^{4(U_0+t\alpha)}}_{bounded} \underbrace{|Dw_0|^2}_{\sim\rho^4}$ is bounded, which is uniformly integrable. On B_δ, $\frac{1}{\rho^4} \underbrace{\alpha^2}_{\sim\log^2 r} \underbrace{e^{4(U_0+t\alpha)}}_{\sim r^{4(1+t)}} \underbrace{|Dw_0|^2}_{\sim\rho^4 r^{-6}} \le C(\log^2 r) r^{-2}$ which is also uniformly integrable.

Combing these together, we get the convexity of the reduced energy $\mathcal{I}$ along geodesic paths. □

Let us check that the first variation at (U_0, w_0) is zero.

Lemma 4.10. *We have* $\frac{d}{dt}\Big|_{t=0} \mathcal{I}(U_t, w_t) = 0$.

Proof. By taking $\mu \ll \epsilon$ and $\lambda \ll \delta$,

$$\frac{d}{dt}\Big|_{t=0} \mathcal{I}(U_t, w_t) = \underbrace{\int_{\Omega_{\lambda,\mu}}}_{I_1} + \underbrace{\int_{\mathcal{A}_{\lambda,\mu}}}_{I_2} [2\langle DU_0, DU_0'\rangle + 4U_0' \frac{e^{4U_0}}{\rho^4} |Dw_0|^2 + 2\frac{e^{4U_0}}{\rho^4} \langle Dw_0, Dw_0'\rangle] dx.$$

Using integration by parts and the fact that (U_0, w_0) satisfies the Euler-Lagrange equation for $\mathcal{I}$ and that $(U_0', w_0') = (\alpha, 0)$ in a neighborhood of $\mathcal{A}_{\lambda,\mu}$, we have

$$I_1 = \int_{\partial\mathcal{A}_{\lambda,\mu}} 2\frac{\partial}{\partial n} U_0 \cdot \alpha.$$

Now separating $\mathcal{A}_{\lambda,\mu} = B_\lambda \cup \mathcal{C}_{\lambda,\mu}$,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{I}(U_t, w_t) = & \underbrace{\int_{\partial\mathcal{C}_{\lambda,\mu}} 2\frac{\partial}{\partial n} U_0 \cdot \alpha d\sigma}_{I_1} + \underbrace{\int_{\mathcal{C}_{\lambda,\mu}} 2\langle DU_0, D\alpha\rangle + 4\alpha \frac{e^{4U_0}}{\rho^4} |Dw_0|^2 dx}_{I_2} \\ & + \underbrace{\int_{\partial B_\lambda} 2\frac{\partial}{\partial n} U_0 \cdot \alpha d\sigma}_{I_3} + \underbrace{\int_{B_\lambda} 2\langle DU_0, D\alpha\rangle + 4\alpha \frac{e^{4U_0}}{\rho^4} |Dw_0|^2 dx}_{I_4}. \end{aligned}$$

Since the equation above is always true for all $\mu \ll \epsilon$ and $\lambda \ll \delta$, we can take a limit by first letting $\mu \to 0$, and then $\lambda \to 0$. For fixed $\lambda \ll \delta$, the integrands in both I_1 and I_2 are bounded, so $I_1, I_2 \to 0$ as $\mu \to 0$. Now $\underbrace{\frac{\partial}{\partial n} U_0}_{\sim r^{-1}} \cdot \underbrace{\alpha}_{\sim\log r} \underbrace{d\sigma}_{\sim r^2 d\sigma_0} \sim r \log r d\sigma_0 \to 0$[11] as $\lambda \to 0$, hence $I_3 \to 0$. I_4 converges to 0 as $\lambda \to 0$, since both DU_0 and $D\alpha$ are L^2 integrable, and $\frac{1}{\rho^4} \underbrace{\alpha}_{\sim\log r} \underbrace{e^{4(U_0)}}_{\sim r^4} \underbrace{|Dw_0|^2}_{\sim\rho^4 r^{-6}} \sim (\log r) r^{-2}$ is also uniformly integrable. We have finished the proof of the lemma. □

[11] $d\sigma_0$ is the volume form on standard sphere.

Proof of Theorem 4.2: Combining Lemma 4.9 and Lemma 4.10, integrating as in Section 3, and using the Sobolev inequality(see [9]), we can get:

$$\begin{aligned}\mathcal{I}(U_\delta, w_{\delta,\epsilon}) - \mathcal{I}(U_0, w_0) &\geq \frac{1}{4}\int_{\mathbb{R}^3} \left|D[d_{-1}\big((U_\delta, w_{\delta,\epsilon}), (U_0, w_0)\big)]\right|^2 dx \\ &\geq C\{\int_{\mathbb{R}^3} d_{-1}^6\big((U_\delta, w_{\delta,\epsilon}), (U_0, w_0)\big)dx\}^{1/3}.\end{aligned}$$

We will first take the limit as $\epsilon \to 0$, and then $\delta \to 0$, then the left hand side will converge to $\mathcal{I}(U, w) - \mathcal{I}(U_0, w_0)$ by Proposition 4.4. Now we will show that the right hand side converges to $\{\int_{\mathbb{R}^3} d_{-1}^6\big((U, w), (U_0, w_0)\big)dx\}^{1/3}$. By the triangle inequality, it suffices to show the following.

Lemma 4.11. *We have* $\int_{\mathbb{R}^3} d_{-1}^6\big((U_\delta, w_{\delta,\epsilon}), (U, w)\big)dx \to 0.$

Proof. In fact,

$$\begin{aligned}d_{-1}\big((U_\delta, w_{\delta,\epsilon}), (U, w)\big) &\leq d_{-1}\big((U_\delta, w_{\delta,\epsilon}), (U, w_{\delta,\epsilon})\big) + d_{-1}\big((U, w_{\delta,\epsilon}), (U, w)\big) \\ &= 2|U - U_\delta| + 2\frac{e^{2U}}{\rho^2}|w - w_{\delta,\epsilon}|.\end{aligned}$$

Now we need to consider,

$$\int_{\mathbb{R}^3} (U - U_\delta)^6 dx \sim \int_{\mathbb{R}^3 \setminus B_{1/\delta}} \underbrace{(U - U_0)^6}_{\sim o(r^{-3})} dx,$$

which converges to 0 as $\delta \to 0$. Using asymptotic estimates as before,

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{e^{12U}}{\rho^{12}}(w - w_{\delta,\epsilon})^6 dx \sim &\int_{\mathbb{R}^3 \setminus B_{1/\delta}} \frac{1}{\rho^{12}} \underbrace{e^{12U}}_{\leq 2} \underbrace{(w - w_0)^6}_{\sim \rho^{18} r^{-6\lambda}} dx + \int_{\mathcal{C}_{\delta,\epsilon}} \frac{1}{\rho^{12}} \underbrace{e^{12U}}_{\leq C} \underbrace{(w - w_0)^6}_{\sim C\rho^{18}} dx \\ &+ \int_{B_{2\delta}} \frac{1}{\rho^{12}} \underbrace{e^{12U}}_{\sim r^{24}} \underbrace{(w - w_0)^6}_{\sim \rho^{18} r^{6\lambda - 36}} dx.\end{aligned}$$

The second term is $\sim \epsilon^8$, and converges to 0, when δ fixed. The first term is $\sim \delta^{6(\lambda - 3/2)}$, which converges to 0 for $\lambda > 3/2$ when $\delta \to 0$. The third term is $\sim \delta^{6\lambda - 3}$, and this converges to 0 as $\delta \to 0$. □

5 Einstein Maxwell case

Motivated by the work of P. Chruściel and J. Costa [3] and G. Weinstein [11], we will extend the convexity and Sobolev bound to another renormalized harmonic energy functional corresponding to the axisymmetric vacuum Einstein/Maxwell equations. For this purpose we

consider the mapping $\tilde{\Psi} = (u, v, \chi, \psi) : \mathbb{R}^3 \to \mathbb{H}^2_{\mathbb{C}}$, where $\mathbb{H}^2_{\mathbb{C}} = \{(u, v, \chi, \psi) \in \mathbb{R}^4\}$ is the complex hyperbolic plane with metric

$$ds^2_{\mathbb{H}_{\mathbb{C}}} = du^2 + e^{4u}(dv + \chi d\psi - \psi d\chi)^2 + e^{2u}(d\chi^2 + d\psi^2).$$

The harmonic energy functional E of $\tilde{\Psi} : \Omega \to \mathbb{H}_{\mathbb{C}}$ is

$$E_\Omega(\tilde{\Psi}) = \int_\Omega |du|^2 + e^{4u}|dv + \chi d\psi - \psi d\chi|^2 + e^{2u}(|d\chi|^2 + |d\psi|^2)dx, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^3$. Writing

$$U = u + \log \rho, \tag{5.2}$$

we can rewrite the above mapping as $\Psi = (U, v, \chi, \psi)$. We are interested in the following functional discussed in [3][6],

$$\mathcal{I}_\Omega(\Psi) = \int_\Omega |DU|^2 + \frac{e^{4U}}{\rho^4}|Dv + \chi D\psi - \psi D\chi|^2 + \frac{e^{2U}}{\rho^2}(|D\chi|^2 + |D\psi|^2)dx, \tag{5.3}$$

where $\Omega \subset \mathbb{R}^3$, and we write $\mathcal{I} = \mathcal{I}_{\mathbb{R}^3}$. Now denote the one form ω by

$$\omega = Dv + \chi D\psi - \psi D\chi \tag{5.4}$$

so that

$$\mathcal{I}_\Omega(\Psi) = \int_\Omega |DU|^2 + \frac{e^{4U}}{\rho^4}|\omega|^2 + \frac{e^{2U}}{\rho^2}(|D\chi|^2 + |D\psi|^2)dx. \tag{5.5}$$

An result similar to (2.5) can be derived by putting (5.2) into (5.5) and using integration by parts together with the fact that $\log \rho$ is harmonic on $\mathbb{R}^3 \setminus \Gamma$,

$$\mathcal{I}_\Omega(\Psi) = E_\Omega(\tilde{\Psi}) + \int_{\partial\Omega} \frac{\partial \log \rho}{\partial n}(2U + \log \rho)d\sigma, \tag{5.6}$$

where Ω is a compact region in $\mathbb{R}^3 \setminus \Gamma$, and n is the unit outer normal of $\partial\Omega$.

In fact, the extreme Kerr-Newman solution of the Einstein/Maxwell equations is a local critical point of $\mathcal{I}$[12]. The extreme Kerr-Newman solution is determined by a map $\tilde{\Psi}_0 = (u_0, v_0, \chi_0, \psi_0)$, or equivalently $\Psi_0 = (U_0, v_0, \chi_0, \psi_0)$ with $U_0 = u_0 + \log \rho$, which is given (see [5], [11]) as

$$\begin{aligned}
u_0 &= -\frac{1}{2} \log \left[(\tilde{r}^2 + a^2 + \frac{a^2 \sin^2 \theta (2m\tilde{r} - q^2)}{\Sigma}) \sin^2 \theta\right] \\
v_0 &= ma \cos \theta (3 - \cos^2 \theta) - \frac{a(q^2 \tilde{r} - ma^2 \sin^2 \theta) \cos \theta \sin^2 \theta}{\Sigma} \\
\chi_0 &= -\frac{qa\tilde{r} \sin^2 \theta}{\Sigma} \\
\psi_0 &= q\frac{(\tilde{r}^2 + a^2) \cos \theta}{\Sigma},
\end{aligned} \tag{5.7}$$

[12] See [11] for details.

where $m^2 = a^2 + q^2$, and

$$\tilde{r} = r + m,\ \Sigma = \tilde{r}^2 + a^2 \cos^2 \theta.$$

Here m is the ADM mass, $J = ma$ the angular-momentum, and q the electric charge.

We are interested in the class of mappings $\Psi = (U, v, \chi, \psi)$ with finite reduced energy $\mathcal{I}(\Psi) < \infty$, which physically corresponds to axisymmetric initial data sets for the Einstein/Maxwell equations[13]. Here we will consider a class of maps which are variations from extreme Kerr-Newman map. Denote the difference $(\Delta U, \Delta v, \Delta \chi, \Delta \psi)$ by

$$\Delta U = U - U_0,\ \Delta v = v - v_0,\ \Delta \chi = \chi - \chi_0,\ \Delta \psi = \psi - \psi_0. \tag{5.8}$$

Motivated by the setting in [7], we consider the following restrictions on $(\Delta U, \Delta v, \Delta \chi, \Delta \psi)$,

$$\begin{aligned} &\Delta U \in H^1_0(\mathbb{R}^3),\ (\Delta U)_+ \in L^\infty(\mathbb{R}^3), \\ &(\omega - \omega_0) \in L^2_{0,\frac{e^{2U_0}}{\rho^2}}(\mathbb{R}^3), \\ &\Delta \chi, \Delta \psi \in H^1_{0,\frac{e^{U_0}}{\rho}}(\mathbb{R}^3),\ \frac{e^{U_0}}{\rho}\Delta \chi, \frac{e^{U_0}}{\rho}\Delta \psi \in L^\infty(\mathbb{R}^3), \end{aligned} \tag{5.9}$$

where $(\Delta U)_+$ denotes the positive part of Ụ, and $H^1_{0,X}(\mathbb{R}^3)$ is defined in (1.8).

Remark 5.1. *This is a relatively restrictive requirement. We put it here in order to show a simple and direct proof compared to that in the next section.*

Lemma 5.2. *Under condition (5.9), $\mathcal{I}(\Psi)$ is finite.*

Proof. Since $(\Delta U)_+ \in L^\infty(\mathbb{R}^3)$, we know that $\frac{e^U}{\rho} \leq C\frac{e^{U_0}}{\rho}$, so $H^1_{0,\frac{e^{U_0}}{\rho}}(\mathbb{R}^3) \subset H^1_{0,\frac{e^U}{\rho}}(\mathbb{R}^3)$ and $H^1_{0,\frac{e^{2U_0}}{\rho^2}}(\mathbb{R}^3) \subset H^1_{0,\frac{e^{2U}}{\rho^2}}(\mathbb{R}^3)$. The lemma now follows. □

Lemma 5.3. *Under condition (5.9), $\Delta v \in H^1_{0,X}(\mathbb{R}^3)$, where X is a smooth function defined on $\mathbb{R}^3 \setminus \Gamma$, with $X = \frac{e^{U_0}}{\rho}$ in a neighborhood of Γ, and $X = \frac{e^{2U_0}}{\rho^2}$ elsewhere near ∞.*

Proof. We compute

$$\begin{aligned} \omega &= (Dv + \chi D\psi - \psi D\chi) \\ &= Dv_0 + D\Delta v + (\chi_0 + \Delta\chi)D(\psi_0 + \Delta\psi) - (\psi_0 + \Delta\psi)D(\chi_0 + \Delta\chi) \\ &= \omega_0 + (D\Delta v + \Delta\chi D\Delta\psi - \Delta\psi D\Delta\chi) \\ &+ (\Delta\chi D\psi_0 - \Delta\psi D\chi_0 + \chi_0 D\Delta\psi - \psi_0 D\Delta\chi). \end{aligned}$$

Therefore

$$\begin{aligned} D\Delta v = (\omega - \omega_0) &- (\Delta\chi D\Delta\psi - \Delta\psi D\Delta\chi) - (\Delta\chi D\psi_0 - \Delta\psi D\chi_0) \\ &- \chi_0 D\Delta\psi + \psi_0 D\Delta\chi. \end{aligned}$$

[13]See [11] for initial data equation, and see [3][6] for the relation between Ψ and initial data.

In fact, from (5.9) and the asymptotic behavior of Ψ_0 (See Appendix A in [6]), all terms except for $\psi_0 D\Delta\chi$ lie in $L^2_{0,\frac{e^{2U_0}}{\rho^2}}(\mathbb{R}^3)$, which are also in $L^2_{0,\frac{e^{U_0}}{\rho}}(\mathbb{R}^3)$ near the axis Γ, where $\frac{e^{U_0}}{\rho} \leq \frac{e^{2U_0}}{\rho^2}$. The last term $\psi_0 D\Delta\chi$ lies in $L^2_{0,\frac{e^{U_0}}{\rho}}(\mathbb{R}^3)$ since ψ_0 is bounded, so it also lies in $L^2_{0,\frac{e^{2U_0}}{\rho^2}}(\mathbb{R}^3)$ as $\frac{e^{2U_0}}{\rho^2} \leq \frac{e^{U_0}}{\rho}$ elsewhere near ∞. Thus we have finished the proof. □

Theorem 5.4. *$\mathcal{I}(\Psi)$ has a global minimum at the Extreme Kerr-Newman Ψ_0, when $(\Psi - \Psi_0)$ satisfies conditions (5.9), i.e.*

$$\mathcal{I}(\Psi) \geq \mathcal{I}(\Psi_0). \tag{5.10}$$

Furthermore, we have the gap bound,

$$\mathcal{I}(\Psi) - \mathcal{I}(\Psi_0) \geq C\{\int_{\mathbb{R}^3} d^6_{\mathbb{H}_\mathbb{C}}(\Psi, \Psi_0)\}^{1/3}. \tag{5.11}$$

Proof. The key point is that we can approximate ΔU, Δv, $(\Delta\chi, \Delta\psi)$ by compactly supported smooth functions in $C^\infty_c(A_{R,\epsilon})$ and $C^\infty_c(\Omega_{R,\epsilon})$ (see section 2.2 for definition) under $H^1_0(\mathbb{R}^3)$, $H^1_{0,X}(\mathbb{R}^3)$, $H^1_{0,\frac{e^{U_0}}{\rho}}(\mathbb{R}^3)$ norms respectively. Then the remainder of the proof is exactly the same as in the proof of Theorem 1.1 except that we use (5.6) instead of (2.5). We will address the details in next section. □

6 Extension to Chruściel-Costa data

Now we will extend the above result to a more general setting coming from physical asymptotic conditions described in [3], [6]. In fact, we can handle weaker asymptotic conditions than [3], [6]; for example, we need only assume $h, E, B = O_{k-1}(\frac{1}{r^\lambda})$ with $\lambda > \frac{3}{2}$[14], where h, E and B are the second fundamental form, electric, and magnetic fields respectively.

In the notation described in the next section, we can state the main theorem which shows that Ψ_0 (extreme Kerr-Neuman) is the global minimum point of the reduced energy.

Theorem 6.1. *For $k \geq 6$, $\mathcal{I}(\Psi)$ is bounded from below by the corresponding value of the extreme Kerr-Newman map (5.7), i.e. for any map $\Psi = (U, v, \chi, \psi)$ satisfying (4.4)(4.6)(6.3)(6.4) (6.5) and (6.6) we have*

$$\mathcal{I}(\Psi) \geq \mathcal{I}(\Psi_0). \tag{6.1}$$

Furthermore, we have the gap inequality,

$$\mathcal{I}(\Psi) - \mathcal{I}(\Psi_0) \geq C\{\int_{\mathbb{R}^3} d^6_{\mathbb{H}_\mathbb{C}}(\Psi, \Psi_0)dx\}^{1/3}. \tag{6.2}$$

[14]Compare to [3][6], where they assume $h = O(\frac{1}{r^\beta})$ with $\beta > \frac{5}{2}$, $E, B = O(\frac{1}{r^{1+\gamma}})$ with $\gamma > \frac{3}{4}$.

6.1 Asymptotic behavior

We first describe the singular behavior of Ψ. From [2], we can assume U satisfies (4.4) and (4.6). From the asymptotic flatness conditions (see [3], [6]) for corresponding initial data sets, we can assume the decay rate of (ω, χ, ψ) at ∞ is

$$|\omega| = \rho^2 O(r^{-\lambda});\ |D\chi|, |D\psi| = \rho O(r^{-\lambda}),\ r \to \infty, \tag{6.3}$$

where we assume the decay rate of electric and magnetic fields is $O(r^{-\lambda})$[15]. Now using an inversion near 0,

$$|\omega| = \rho^2 O(r^{\lambda-6});\ |D\chi|, |D\psi| = \rho O(r^{\lambda-4}),\ r \to 0. \tag{6.4}$$

Near the axis $\Gamma = \{\rho = 0\}$, we can assume that,

$$|\omega| = O(\rho^2);\ |D\chi|, |D\psi| = O(\rho),\ \rho \to 0,\ \delta \le r \le 1/\delta. \tag{6.5}$$

Furthermore, we assume that the data corresponding to Ψ has the same angular momentum and electric-magnetic charge as the extreme Kerr-Neuman data given by Ψ_0, i.e. they have the same value restricted to the axis $\Gamma = \mathcal{A}_1 \cup \mathcal{A}_2$[16],

$$v|_\Gamma = v_0|_\Gamma = \begin{cases} -2ma, & \text{on } \mathcal{A}_1 \\ 2ma, & \text{on } \mathcal{A}_2 \end{cases},\ \chi|_\Gamma = \chi_0|_\Gamma = 0,\ \psi|_\Gamma = \psi_0|_\Gamma = \begin{cases} -q, & \text{on } \mathcal{A}_1 \\ q, & \text{on } \mathcal{A}_2 \end{cases}. \tag{6.6}$$

Now let us derive more asymptotic conditions on the data. Using the boundary behavior (6.6) and integrating (6.3) along a line perpendicular to Γ,

$$|\chi| = \rho^2 O(r^{-\lambda}),\ |\psi| = const + \rho^2 O(r^{-\lambda}) = O(r^{-\lambda+2}),\ r \to \infty. \tag{6.7}$$

Similarly integrating (6.4),

$$|\chi| = \rho^2 O(r^{\lambda-4}),\ |\psi| = const + \rho^2 O(r^{\lambda-4}) = O(r^{\lambda-2}),\ r \to 0. \tag{6.8}$$

Near the axis we can integrate (6.6)

$$|\chi| = O(\rho^2),\ |\psi| = O(1),\ \rho \to 0,\ \delta \le r \le 1/\delta. \tag{6.9}$$

Now combining with (5.4)(6.3)(6.4) and (6.7)(6.8)(6.9), we have

$$|Dv| \le |\omega| + |\chi D\psi - \psi D\chi| = \rho^2 O(r^{-\lambda}) + \rho O(r^{-2\lambda+2}) = \rho O(r^{-\lambda+1}),\ r \to \infty. \tag{6.10}$$

$$|Dv| \le |\omega| + |\chi D\psi - \psi D\chi| = \rho^2 O(r^{\lambda-6}) + \rho O(r^{2\lambda-6}) = \rho O(r^{\lambda-5}),\ r \to 0. \tag{6.11}$$

$$|Dv| \le |\omega| + |\chi D\psi - \psi D\chi| = O(\rho^2) + O(\rho) = O(\rho),\ \rho \to 0,\ \delta \le r \le 1/\delta. \tag{6.12}$$

[15] Compare with (2.3) in [6].

[16] See discussion on page 4 in [6]. $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined in section 4.1.

Remark 6.2. *Let us quickly review the integrability of* $\mathcal{I}(\Psi)$. *The* $|DU|^2$ *term is the same as in the vacuum case, and the term* $\frac{e^{4U}}{\rho^4}|\omega|^2$ *is the same as* $\frac{e^{4U}}{\rho^4}|dw|^2$ *in Remark 4.3. Now for* (χ, ψ), *near* ∞, $\frac{e^{2U}}{\rho^2}(|D\chi|^2 + |D\psi|^2) = O(r^{-2\lambda})$ *is integrable for* $\lambda > \frac{3}{2}$. *Near 0,* $\frac{e^{2U}}{\rho^2}(|D\chi|^2 + |D\psi|^2) = O(r^{2\lambda-4})$ *is also integrable.*

Now let us also list the asymptotic behavior of Ψ_0

$$|\omega_0| = \rho^2 O(r^{-3}),\ |D\chi_0| = \rho O(r^{-3}), |D\psi_0| = \rho O(r^{-2}),\ \chi_0 = \rho^2 O(r^{-3}),\ \psi_0 = O(1),\ r \to \infty. \tag{6.13}$$

$$|\omega_0| = \rho^2 O(r^{-3}),\ |D\chi_0|, |D\psi_0| = \rho O(r^{-2}),\ \chi_0 = \rho^2 O(r^{-2}),\ \psi_0 = O(1),\ r \to 0. \tag{6.14}$$

Here the behavior of ω *is gotten by direct calculations based on (5.7), and other calculations can be found in Appendix A in [6].*

6.2 Cut and paste argument

Given $\Psi = (U, v, \chi, \psi)$ as in Theorem 6.1, we approximate $\mathcal{I}(\Psi)$ again by cutting and pasting Ψ to Ψ_0 near ∞, and then cutting and pasting (v, χ, ψ) to (v_0, χ_0, ψ_0) near 0 and axis Γ.

Propostion 6.3. *Under conditions (4.4)(4.6)(6.3)(6.4)(6.5) and (6.6) for* $\Psi = (U, v, \chi, \psi)$, *for any small* $c_0 > 0$, *we can find* $\Psi_{\delta,\epsilon} = (U_\delta, v_{\delta,\epsilon}, \chi_{\delta,\epsilon}, \psi_{\delta,\epsilon})$ *for small* $\epsilon \ll \delta \ll 1$, *such that:*

$$U_\delta \equiv U,\ r < 1/\delta;\ (v_{\delta,\epsilon}, \chi_{\delta,\epsilon}, \psi_{\delta,\epsilon}) \equiv (v, \chi,\ \psi),\ \rho > \sqrt{\epsilon},\ 2\delta < r < 1/\delta,$$

$$(U_\delta, v_{\delta,\epsilon}, \chi_{\delta,\epsilon}, \psi_{\delta,\epsilon}) = (U_0, v_0, \chi_0, \psi_0),\ r > 2/\delta,$$

$$(v_{\delta,\epsilon}, \chi_{\delta,\epsilon}, \psi_{\delta,\epsilon}) \equiv (v_0, \chi_0, \psi_0),\ x \in B_\delta \cup \mathcal{C}_{\delta,\epsilon},$$

where $\mathcal{C}_{\delta,\epsilon}$ *is defined in (4.24), and*

$$|\mathcal{I}(\Psi) - \mathcal{I}(\Psi_{\delta,\epsilon})| < c_0.$$

As in the vacuum case, we can achieve this approximation is three steps. Now we will sketch the proof. First define

$$\Psi^1_\delta = \Psi_0 + \varphi^1_\delta(\Psi - \Psi_0),$$

where φ^1_δ is defined in (4.21). Then $\Psi^1_\delta = \Psi_0$ outside $B_{2/\delta}$.

Lemma 6.4. $\lim_{\delta\to 0} \mathcal{I}(\Psi^1_\delta) = \mathcal{I}(\Psi)$.

Proof. By comparing to the proof of lemma 4.5, the only difference from that case is to show

$$\int_{1/\delta<r<2/\delta} \frac{e^{4U^1_\delta}}{\rho^4}|\omega^1_\delta|^2 dx \to 0,$$

where (by (2.16) in [6])

$$\omega_\delta^1 = \varphi_\delta^1\omega + (1-\varphi_\delta^1)\omega_0 + \underbrace{D\varphi_\delta^1(v-v_0)}_{\sim\delta\rho^2 r^{-\lambda+1}} + \underbrace{D\varphi_\delta^1(\chi_0\psi-\psi_0\chi)}_{\sim\delta\rho^2 r^{-\lambda}}$$
$$+ \varphi_\delta^1(1-\varphi_\delta^1)\underbrace{\{(\psi-\psi_0)D(\chi-\chi_0)-(\chi-\chi_0)D(\psi-\psi_0)\}}_{\sim\rho^2 r^{-2\lambda+1}}.$$

The asymptotic behavior comes from (6.10)(6.7)(6.6)(6.3) and those of Extreme-Kerr coming from Appendix A in [6]. Convergence follows from the asymptotics. □

Now we can assume $\Psi = \Psi_0$ outside $B_{2/\delta}$. Define

$$(v_\delta, \chi_\delta, \psi_\delta) = (v_0, \chi_0, \psi_0) + \varphi_\delta(v-v_0, \chi-\chi_0, \psi-\psi_0),$$

where φ_δ is defined in (4.22). Then $(v_\delta, \chi_\delta, \psi_\delta) = (v_0, \chi_0, \psi_0)$ in B_δ. Let $\Psi_\delta = (U, v_\delta, \chi_\delta, \psi_\delta)$.

Lemma 6.5. *We have* $\lim_{\delta\to 0}\mathcal{I}(\Psi_\delta) = \mathcal{I}(\Psi)$.

Proof. By comparing to the proof of lemma 4.6, the different term we need to handle is,

$$\int_{\delta<r<2\delta}\frac{e^{4U}}{\rho^4}|\omega_\delta|^2 dx \to 0,$$

while

$$\omega_\delta = \varphi_\delta\omega + (1-\varphi_\delta)\omega_0 + \underbrace{D\varphi_\delta(v-v_0)}_{\sim(1/\delta)\rho^2 r^{\lambda-5}} + \underbrace{D\varphi_\delta(\chi_0\psi-\psi_0\chi)}_{\sim(1/\delta)\rho^2 r^{\lambda-4}}$$
$$+ \varphi_\delta(1-\varphi_\delta)\underbrace{\{(\psi-\psi_0)D(\chi-\chi_0)-(\chi-\chi_0)D(\psi-\psi_0)\}}_{\sim\rho^2 r^{2\lambda-7}},$$

where the asymptotics come from (6.11)(6.8)(6.4)(6.6). Convergence follows from the asymptotics and the fact that $\frac{e^{4U}}{\rho^4} \sim \frac{r^8}{\rho^4}$. □

Remark 6.6. *The reason we can improve to* $\lambda > \frac{3}{2}$ *(weaker than [3], [6]) is that* $e^{4U} \sim r^8$ *by (4.6) is faster than* $e^{4U_0} \sim r^4$ *by (4.19), while we did not cut* U *off near* 0.

Now we can assume furthermore that $(v, \chi, \psi) = (v_0, \chi_0, \psi_0)$ in B_δ. Define

$$(v_\epsilon, \chi_\epsilon, \psi_\epsilon) = (v_0, \chi_0, \psi_0) + \phi_\epsilon(v-v_0, \chi-\chi_0, \psi-\psi_0),$$

with ϕ_ϵ defined in (4.23). Now $(v_\epsilon, \chi_\epsilon, \psi_\epsilon) = (v_0, \chi_0, \psi_0)$ in $\mathcal{C}_{\delta,\epsilon}\cup B_\delta$. Denote $\Psi_\epsilon = (U, v_\epsilon, \chi_\epsilon, \psi_\epsilon)$.

Lemma 6.7. *We have* $\lim_{\epsilon\to 0}\mathcal{I}(\Psi_\epsilon) = \mathcal{I}(\Psi)$.

Proof. By comparing to the proof of lemma 4.8, the additional term we need to handle is,

$$\int_{\mathcal{W}_{\delta,\epsilon}} \frac{e^{4U}}{\rho^4} |\omega_\epsilon|^2 dx \to 0,$$

while

$$\begin{aligned}\omega_\epsilon = \phi_\epsilon \omega + (1-\phi_\epsilon)\omega_0 + \underbrace{D\phi_\epsilon(v - v_0)}_{\sim (1/(\rho \ln \epsilon))\rho^2} + \underbrace{D\phi_\epsilon(\chi_0\psi - \psi_0\chi)}_{\sim (1/(\rho \ln \epsilon))\rho^2} \\ + \phi_\epsilon(1-\phi_\epsilon)\underbrace{\{(\psi-\psi_0)D(\chi-\chi_0) - (\chi-\chi_0)D(\psi-\psi_0)\}}_{\sim \rho^3},\end{aligned}$$

where the asymptotics come from (6.12)(6.9)(6.5)(6.6). Convergence follows from these asymptotics. □

Combining the above three lemmas, we have proven Proposition 6.3.

6.3 Convexity and gap inequality

The proof of Theorem 6.1 is very similar to that in Section 4.3. We will point out the main differences here. By Proposition 6.3, we can first take $(\Delta U, \Delta v, \Delta \chi, \Delta \psi)$ in (5.8) to satisfy: (1) ΔU is compactly supported in $B_{2/\delta}$; (2) $(\Delta v, \Delta\chi, \Delta\psi)$ are compactly supported in $\Omega_{\delta,\epsilon}$, which is defined in (4.26).

Now we can connect $\tilde{\Psi} = (u = U - \log\rho, v, \chi, \psi)$ to $\tilde{\Psi}_0 = (u_0 = U_0 - \log\rho, v_0, \chi_0, \psi_0)$ by a geodesic family $\tilde{\Psi}_t = (u_t, v_t, \chi_t, \psi_t)$ on $(\mathbb{H}^2_{\mathbb{C}}, ds^2_{\mathbb{H}_{\mathbb{C}}})$. Denote $U_t = u_t + \log\rho$. We know that $\Psi_t \equiv \Psi_0$ outside $B_{2/\delta}$. Then $(v_t, \chi_t, \psi_t) \equiv (v_0, \chi_0, \psi_0)$ in a neighborhood of $\mathcal{A}_{\delta,\epsilon}$ (defined in (4.27)). So $U_t = U_0 + t\Delta U$ in a neighborhood of $\mathcal{A}_{\delta,\epsilon}$ as in Section 2. As in Lemma 4.9, we have

Lemma 6.8. *The following inequality holds*

$$\frac{d^2}{dt^2}\mathcal{I}(\Psi_t) \geq 2\int_{\mathbb{R}^3} |D\big(d_{\mathbb{H}_{\mathbb{C}}}(\Psi, \Psi_0)\big)|^2 dx. \tag{6.15}$$

Proof.

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{I}(\Psi_t) &= \frac{d^2}{dt^2}\mathcal{I}_{B_{2/\delta}}(\Psi_t) \\ &= \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\Omega_{\delta,\epsilon}}(\Psi_t)}_{I_1} + \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\mathcal{A}_{\delta,\epsilon}}(\Psi_t)}_{I_2}.\end{aligned} \tag{6.16}$$

Using formula (5.6), the fact that $(\mathbb{H}^2_{\mathbb{C}}, ds^2_{\mathbb{H}_{\mathbb{C}}})$ is negatively curved and (2.4), the first part is calculated as:

$$\begin{aligned} I_1 &= \frac{d^2}{dt^2} E_{\Omega_{\delta,\epsilon}}(\tilde{\Psi}_t) + \frac{d^2}{dt^2}\int_{\partial\Omega_{\delta,\epsilon}\cap\partial\mathcal{A}_{\delta,\epsilon}} \frac{\partial \log\rho}{\partial n}(2(U_0+t\Delta U)+\log\rho)d\sigma \\ &\geq 2\int_{\Omega_{\delta,\epsilon}} |D\big(d_{\mathbb{H}_{\mathbb{C}}}(\Psi,\Psi_0)\big)|^2 dx. \end{aligned} \tag{6.17}$$

Since $d_{\mathbb{H}_{\mathbb{C}}}(\Psi,\Psi_0) = |\Delta U|$ on $\mathcal{A}_{\delta,\epsilon}$, the second part is calculated as:

$$\begin{aligned} I_2 &= \frac{d^2}{dt^2}\int_{\mathcal{A}_{\delta,\epsilon}} |D(U_0+t\Delta U)|^2 + \frac{e^{4(U_0+t\Delta U)}}{\rho^4}|\omega_0|^2 + \frac{e^{2(U_0+t\Delta U)}}{\rho^2}(|D\chi_0|^2+|D\psi_0|^2)dx \\ &= 2\int_{\mathcal{A}_{\delta,\epsilon}} |D\Delta U|^2 + 8(\Delta U)^2\frac{e^{4(U_0+t\Delta U)}}{\rho^4}|\omega_0|^2 + 2(\Delta U)^2\frac{e^{2(U_0+t\Delta U)}}{\rho^2}(|D\chi_0|^2+|D\psi_0|^2)dx \\ &\geq 2\int_{\mathcal{A}_{\delta,\epsilon}} |D\big(d_{\mathbb{H}_{\mathbb{C}}}(\Psi,\Psi_0)\big)|^2 dx. \end{aligned} \tag{6.18}$$

Now the reason that we can take $\frac{d^2}{dt^2}$ into the integral in the second " = " follows from the same idea as in the proof of 4.9, making use of (6.14). For example, $\underbrace{(\Delta U)^2\frac{e^{4(U_0+t\Delta U)}}{\rho^4}|\omega_0|^2}_{\sim(\log r)^2\frac{r^{4(1+t)}}{\rho^4}\rho^4 r^{-6}} \sim$ $(\log r)^2 r^{-2}$ is uniformly integrable near 0. Other terms follow similarly. We have proven the lemma. ☐

Using the idea in Lemma 4.10, while using the fact that Ψ_0 satisfies the Euler-Lagrange equation for $\mathcal{I}$, we can easily get the following result. We omit the proof here since it is almost the same as Lemma 4.10.

Lemma 6.9. *At $t=0$ we have $\frac{d}{dt}\big|_{t=0}\mathcal{I}(\Psi_t) = 0$.*

Proof of Theorem 6.1: The proof follows exactly the same idea as the proof of Theorem 4.2 by using Proposition 6.3, Lemma 6.8, and Lemma 6.9. We leave details to the reader.

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